Solving PDEs using Continuous Time Markov Chains

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The continuous model

SDE

- A time homogeneous diffusion process

$$dX_t = b(X_t)dW_t$$

- We are normally interested in calculating the expected value of functions of X_T

$$u(t, x; T, f) = \mathbb{E}\left[f(X_T)|X_t = x\right]$$

- u satisfies a PDE (backward)

$$\partial_t u + \frac{1}{2}b^2\partial_{xx}u = 0$$
$$u(T, x) = f(x)$$

The semigroup

SDE

- An SDE generates a *semigroup*

$$[P(s) \circ f](x) = \mathbb{E}[f(X_T)|X_{T-s} = x]$$
$$P(0) = I$$
$$P(t+s) = P(t) \circ P(s)$$

- with well known *infinitesimal generator*

$$[A \circ f](x) = \lim_{s \to 0^+} \frac{[P(s) \circ f](x) - f(x)}{s}$$
$$= \frac{1}{2} b^2(x) \partial_{xx} f(x)$$

- and *formal* solution

$$\partial_s P(s) = A \circ P(s)$$
$$P(s) \circ f = e^{sA} \circ f = \sum_{k=0}^{\infty} \frac{s^k}{k!} A^k \circ f$$

Adjoint semigroup

The forward equation

There is an other semigroup: P^* the *adjoint* of P, defined by

$$\langle P(s) \circ f, g \rangle = \langle f, P^*(s) \circ g \rangle \in \mathbb{R}$$

which acts like an *integration by parts*.
P^* is indeed a semigroup with infinitesimal generator A^*

$$\left[A^* \circ g\right](y) = \frac{1}{2} \partial_{yy} \left(b^2(y) g(y) \right)$$

- P brings functions *backward* (via conditional expectation)
- P^* moves densities *forward* (by simulation)

Forward and backward equations

Both semigroups have wide applications in Finance

- Going backward to price a deal

$$\partial_t u + \frac{1}{2} b^2 \partial_{xx} u = 0$$
$$u(T, x) = f(x)$$

- Going forward to evolve the density

$$\partial_t u - \frac{1}{2} \partial_{yy}(b^2 u) = 0$$
$$u(0, y) = \delta_{x_0}$$

The 2 equations look *similar* enough to be handled by the same solver (even more so, if the coefficients are not state dependent).

Finite difference solvers

- Replace all derivatives with *finite difference approximations* on a grid
- The solution is obtained via a *linear* system
- The system is built *row by row*
- What about boundary conditions?
  Need to complement the system with *exogenously* given equations that hold on the first and last points of the grid:
  1. Ghost point method
  2. Absorption
  3. Linear solution
  4. Known value
  5. Zero probability

  (They end up in the first and last *rows*)

The linear system

In both cases we end up solving a system like

$$\mathbf{u}_{i+1} - \mathbf{u}_i = hA\mathbf{u}_{i+1}$$

with a matrix qualitatively similar to

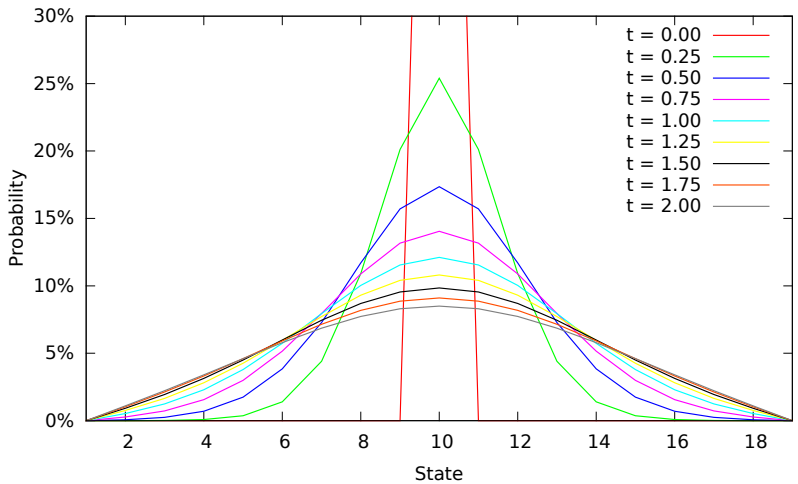
$$A \sim \begin{bmatrix} -x & x & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & \\ & & & y & -y & \end{bmatrix}$$

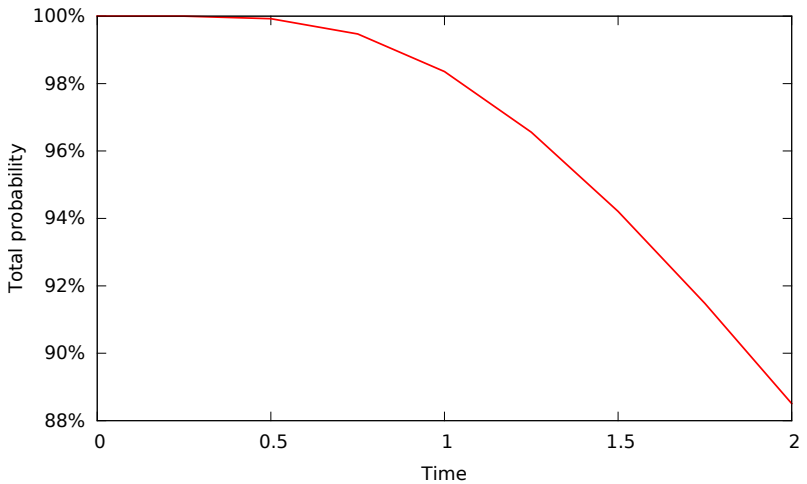
Although this is a *good choice* for the backward equation, it can cause a lot of *issues* in the forward.

Forward equation

Brownian motion

Numerical solution of the forward equation for a Brownian motion (19 points, $\Delta_x = 0.3$, exact matrix exponential)





The loss of mass is due to bad boundary conditions (zero probability). We cannot simply force the solution to be 0.

One step backward

- Let's rewind and start over
- Discretise the process (once!), not the equations (twice!)

The discrete model

Continuous Time Markov Chain

- A *CTMC* is a time homogeneous pure jump process (X_t) fully characterised by the transition *rates* between states
- $\mathcal{X}$ is the state space (finite), $X_t \in \mathcal{X} = \{x_1, \cdots, x_N\}$
- γ_{ij} is the jump intensity from x_i to x_j (Poisson)

$$q_{ij}(h) = \mathbb{P}\left[X_{t+h} = x_j | X_t = x_i\right] = \delta_{ij} + \gamma_{ij} h + o(h)$$

- $\Gamma = (\gamma_{ij})$ is called the transition *rate* matrix
- $Q(h) = (q_{ij}(h))$ is the transition matrix (i.e. probabilities)

The semigroup

CTMC

A CTMC generates a semigroup as well

$$\begin{aligned} \mathbf{z} &\in \mathbb{R}^N \\ P(s) \circ \mathbf{z} &= \mathbb{E}\left[\mathbf{z}(X_T) | X_{T-s}\right] \\ &= \sum_j q_{ij}(s)\mathbf{z}_j = Q(s)\mathbf{z} \end{aligned}$$

with generator

$$\begin{aligned} A \circ \mathbf{z} &= \lim_{s \to 0^+} \frac{Q(s)\mathbf{z} - \mathbf{z}}{s} \\ &= \lim_{s \to 0^+} \frac{\mathbf{z} + s\Gamma\mathbf{z} + o(s) - \mathbf{z}}{s} \\ &= \Gamma\mathbf{z} \end{aligned}$$

CTMC

Transition rate matrix

- γ_{ij} is the intensity of *arriving* to x_j from x_i
- $-\gamma_{ii}$ is the intensity of *leaving* x_i
- since the process leaving x_i must land somewhere, we have

$$\left.\begin{aligned} \gamma_{ij} &\geq 0 \\ \sum_j \gamma_{ij} &= 0 \end{aligned}\right\} \implies \gamma_{ii} = -\sum_{j \neq i} \gamma_{ij} \leq 0$$

- the transition probabilities can be calculated as

$$Q(s) = e^{s\Gamma} \geq 0$$

Transition rate matrix

Example

Jump intensities for the variance process in the Heston model.

$$\begin{bmatrix}
\mathbf{-7.77} & 7.77 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.40 & \mathbf{-2.73} & 2.33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.27 & \mathbf{-1.02} & 0.75 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.84 & \mathbf{-1.01} & 0.17 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.86 & \mathbf{-2.04} & 0.18 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2.69 & \mathbf{-2.87} & 0.18 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3.44 & \mathbf{-3.62} & 0.18 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4.14 & \mathbf{-4.32} & 0.18 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.81 & \mathbf{-5.00} & 0.19 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.26 & \mathbf{-5.26}
\end{bmatrix}$$

1. Off-diagonal positive
2. Diagonal negative
3. Sum by row zero

CTMC

Adjoint semigroup

There is an adjoint semigroup for CTMC as well

$$\begin{aligned}
\mathbf{z}, \mathbf{y} &\in \mathbb{R}^N \\
\langle P(s) \circ \mathbf{z}, \mathbf{y} \rangle &= \langle Q(s)\mathbf{z}, \mathbf{y} \rangle \in \mathbb{R} \\
&= \mathbf{y}' Q(s) \mathbf{z} \\
&= \left\langle \mathbf{z}, Q(s)' \mathbf{y} \right\rangle \\
&= \langle \mathbf{z}, P^*(s) \circ \mathbf{y} \rangle
\end{aligned}$$

The semigroup P^* is the *matrix transpose* of P

$$\begin{aligned}
P^*(s) \circ \mathbf{y} &= e^{s\Gamma'} \mathbf{y} \\
A^* \circ \mathbf{y} &= \Gamma' \mathbf{y}
\end{aligned}$$

Discretisation of an SDE

To discretise an SDE into a CTMC we need to determine

- the state space
  - handle the (new) boundaries (the SDE might not have any)
- the transition intensities
  - the shape of the matrix: full, sparse (do we allow jumps everywhere or just local jumps?)
  - what to preserve in the discrete model

Then we can use the matrix exponential to *solve* the semigroup

The matrix exponential

Exact solution of the discrete process

- The matrix exponential can be computed exactly

$$e^{s\Gamma} = V e^{s\Lambda} V^{-1}$$

  but this is very slow and numerically unstable
- instead we use more tractable formulas to approximate it (Padé approximants)

$$e^{s\Gamma} \sim V R_{(m,n)}(\Lambda) V^{-1}$$
$$e^{x} = R_{(m,n)}(x) + o(x^{m+n})$$

- We are not interested in time discretisation in this context

From SDE to CTMC

- The state space is a finite uniform grid: $x_i = \bar{x} + i\Delta_x$
- The transition rate matrix is multi-tridiagonal: the process can only jump to adjacent states (in 1D, γ_i^- and γ_i^+)
- We try to preserve the first 2 infinitesimal moments
  - SDE (drift-less)

$$\mathbb{E}^x \left[X_h - X_0 \right] = o(h)$$
$$\mathbb{E}^x \left[\left(X_h - X_0 \right)^2 \right] = hb^2(x) + o(h)$$

  - CTMC

$$\mathbb{E}^i \left[X_h - X_0 \right] = h\Delta_x(\gamma_i^+ - \gamma_i^-) + o(h)$$
$$\mathbb{E}^i \left[\left(X_h - X_0 \right)^2 \right] = h\Delta_x^2(\gamma_i^+ + \gamma_i^-) + o(h)$$

CTMC

Choice of coefficients

The system is

$$\begin{cases} \Delta_x(\gamma_i^+ - \gamma_i^-) & = 0 \\ \Delta_x^2(\gamma_i^+ + \gamma_i^-) & = b_i^2 \\ \gamma_i^+, \gamma_i^- & \geq 0 \end{cases}$$

with (familiar) solution

$$\gamma_i^- = \gamma_i^+ = \frac{b_i^2}{2\Delta_x^2}$$

$$\gamma_{ii} = -\gamma_i^- - \gamma_i^+$$

CTMC

Transition rate matrix

We decide to *absorb* the process on the boundaries, $\gamma = 0$ so it cannot jump anywhere.
The transition rate matrix looks like

$$\Gamma = \begin{bmatrix} 0 & 0 & & & \\ \gamma_2^- & \gamma_{22} & \gamma_2^+ & & \\ & \gamma_3^- & \gamma_{33} & \gamma_3^+ & \\ & & \gamma_4^- & \gamma_{44} & \gamma_4^+ \\ & & & 0 & 0 \end{bmatrix}$$

This is a particle that can jumps left or right by one state at a time, and when it reaches the boundaries, it stays there.

Transpose and adjoint

The link between P and P^* is the key to understand how to use the same discretisation for the backward and the forward equation. To calculate an expected value (i.e. a price):

- either: roll back the solution $(s \downarrow 0)$ and select the correct point on the grid (at 0)

$$(P(s) \circ \mathbf{z}) \cdot \mathbf{w}_0 = \mathbf{w}_0' e^{s\Gamma} \mathbf{z}$$

- or: move forward the density $(0 \uparrow s)$ and integrate the solution

$$(P^*(s) \circ \mathbf{w}_0) \cdot \mathbf{z} = \mathbf{z}' e^{s\Gamma'} \mathbf{w}_0$$

We have used the same Γ and the 2 values are the same!

Forward equation

Finite difference

We can now write a (new) finite difference approximation of the forward equation

$$\partial_t u - \frac{1}{2}\partial_{yy}(b^2 u) = 0$$
$$u(0, y) = \delta_{x_0}$$

The discrete solution is

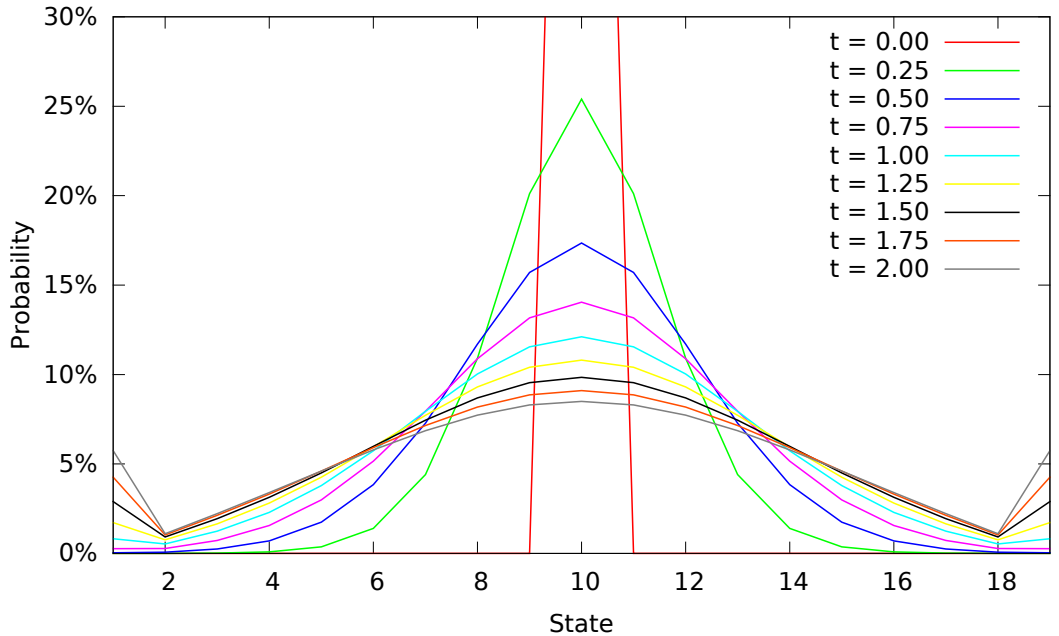
$$\mathbf{u}(t) = e^{s\Gamma'}\mathbb{1}_{x_0}$$

- The linear system is built *column by column*, not *row by row*
- The boundary conditions end up in the first and last *column*

Forward equation

CTMC

Exact solution of the forward equation for a discretised Brownian motion (19 points, $\Delta_x = 0.3$)



- No mass loss
- Probability accumulates at the boundaries (the process is trapped there)

The eigenvalue $\lambda = 0$

Transition rate matrix

Eigenvectors associated to $\lambda = 0$ have interesting properties

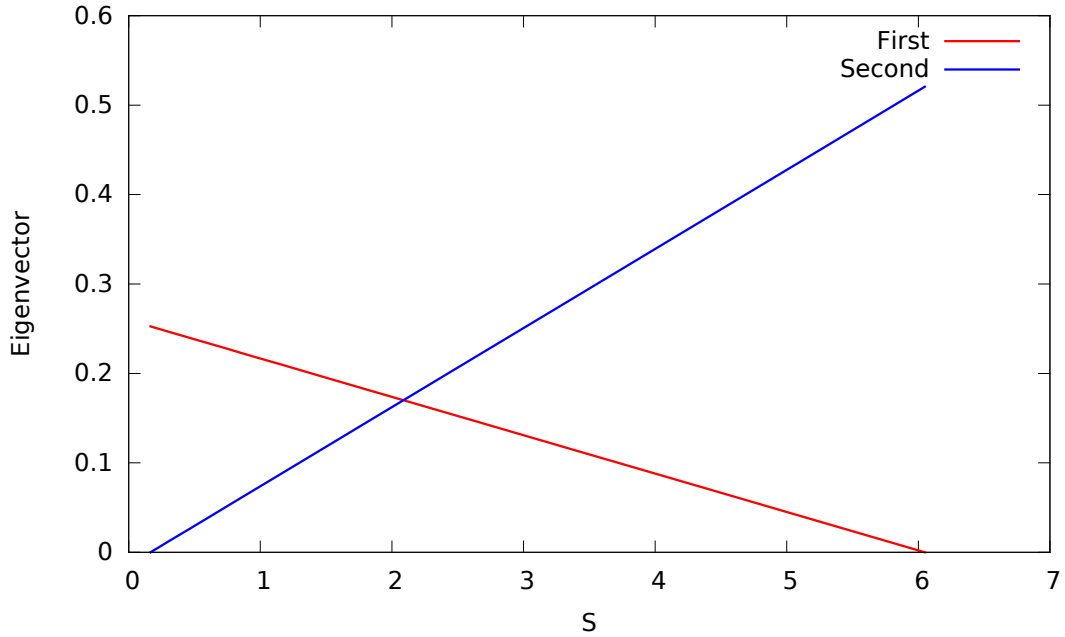
- $\mathbf{1}$ is an eigenvector (zero sum by row)
- if $\mathbf{x}$ is an eigenvector, then X_t is a martingale
- $e^{s\Gamma}$ has the same eigenvectors as Γ, and $e^{s\lambda}$ eigenvalues

$$\Gamma\mathbf{1} = 0\mathbf{1} \implies e^{s\Gamma}\mathbf{1} = e^{s0}\mathbf{1} = \mathbf{1}$$
$$\Gamma\mathbf{x} = 0\mathbf{x} \implies e^{s\Gamma}\mathbf{x} = e^{s0}\mathbf{x} = \mathbf{x}$$

Any time discretisation will preserve this eigenvalue:
by construction we get the correct total mass and expected value.

Lognormal process

Zero eigenvectors



Any payoff which is a linear combination of these 2 eigenvectors

- has constant expected value
- is priced exactly regardless of the time stepping scheme

Introducing a drift

The presence of the drift introduces a minor change to the calculation of the coefficients

$$\mathbb{E}^x \left[X_h - X_0 \right] = ha(x) + o(h)$$

and the new solution is

$$\gamma_i^- = \frac{b_i^2}{2\Delta_x^2} - \frac{a_i}{2\Delta_x}$$
$$\gamma_i^+ = \frac{b_i^2}{2\Delta_x^2} + \frac{a_i}{2\Delta_x}$$

as long as $b^2 \geq |a|\Delta_x$
With a positive drift, it is more likely to jump up: $\gamma_i^+ > \gamma_i^-$

Drift dominated process

What happens when $b^2 < |a|\Delta_x$?
One of the coefficients becomes negative, and the process loses a physical meaning: transition probabilities can go negative.

- Choose a smaller Δ_x: hopefully, the diffusion will dominate
- When solving for coefficients, allow the drift to jump only on one side (*upwind* or *downwind* according to the sign). This is always a valid choice, although potentially less accurate

$$\gamma_i^- = \frac{b_i^2}{2\Delta_x^2} + \frac{a_i^-}{\Delta_x}, \gamma_i^+ = \frac{b_i^2}{2\Delta_x^2} + \frac{a_i^+}{\Delta_x}$$

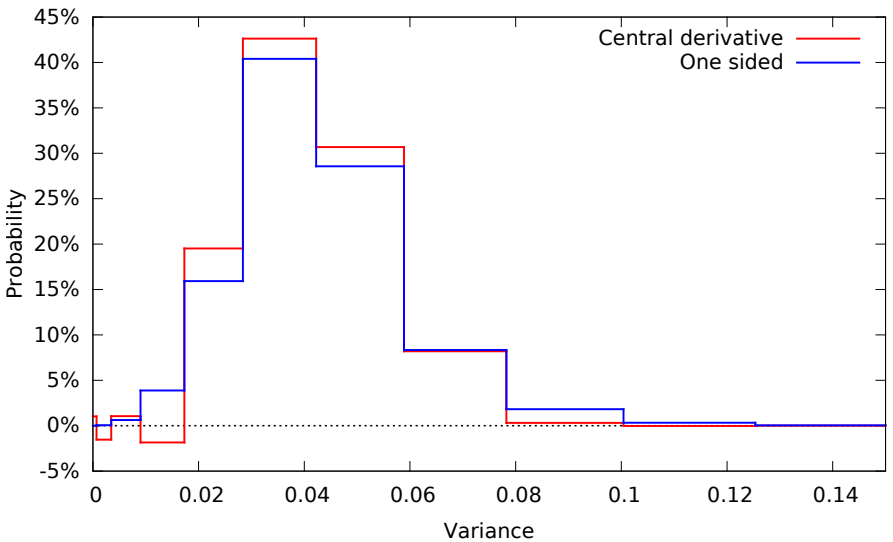
- Use an *exponentially fitted* scheme: increase the volatility to a level compatible with the drift

Drift dominated process

Heston volatility process

Impact of negative off diagonal intensities: strong mean reversion

- $\sigma_0^2 = \sigma_\infty^2 = 20\%^2, \kappa = 120\%, \alpha = 10\%$
- grid: 20 points, $\sigma^2 \in [0, 0.5]$
- T = 2Y



Top left corner of Γ

$$\begin{vmatrix} \boxed{-34} & 34 & 0 & 0 & \cdots \\ \mathbf{-23} & \boxed{19} & 4 & 0 & \cdots \\ 0 & \mathbf{-5} & \boxed{2} & 3 & \cdots \\ 0 & 0 & \mathbf{-1} & \boxed{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

Boundary conditions

Once the particle reaches the boundary, the only degree of freedom is how fast (γ) it is reflected back inside the domain:

- $\gamma = 0$ and it gets trapped there
- $\gamma = \infty$ and it goes back immediately

We can use γ to match the expected value

$$\begin{aligned}\mathbb{E}^1\left[X_h - X_0\right] &= h\Delta_x\gamma + o(h) \\ &= ha_1 + o(h) \\ \gamma &= \frac{a_1}{\Delta_x}\end{aligned}$$

Fine, if the drift points *inward* ($a_1 > 0$).
Otherwise absorption ($\gamma = 0$) seems to be the only choice.

Boundary conditions

Mean reversion

In case of mean reversion (e.g. Heston vol process)

$$d\sigma_t^2 = \kappa(\sigma_\infty^2 - \sigma_t^2)dt + \xi\sigma_t dW_t$$

the drift is *benign*, it always points inward:

$$a_1 = \kappa(\sigma_\infty^2 - \sigma_1^2) \gg 0$$
$$a_N = \kappa(\sigma_\infty^2 - \sigma_N^2) \ll 0$$

and boundary conditions do not cause major headaches.

Boundary conditions

Ghost point

This is a very common technique to handle boundary conditions

- add a new point outside the grid x_0
- assign coefficients in the first row as usual: $\gamma_{1,0}, \gamma_{1,2}$
- impose a linear relationship between the *solution* at the ghost point and in the rest of the domain $\sum_{i=0}^{N} \alpha_i u(x_i) = 0$
- use the same relationship to reallocate the intensity for the non existing point ($\gamma_{1,0}$) to the rest of the grid $-\frac{\alpha_i}{\alpha_0}\gamma_{1,0} \rightarrow \gamma_{1,i}$

Ghost point

Examples

Calculate the intensities ($\hat{\gamma}_{1,0}$) for the ghost point as well (x_0)

$$\boxed{\hat{\gamma}_{1,0} = \frac{b_1^2}{2\Delta_x^2} - \frac{a_1}{2\Delta_x}}, \hat{\gamma}_{1,2} = \frac{b_1^2}{2\Delta_x^2} + \frac{a_1}{2\Delta_x}$$

- linear solution: $\boldsymbol{u(x_0)} - 2u(x_1) + u(x_2) = 0$

$$\gamma_{1,2} = \hat{\gamma}_{1,2} - \hat{\gamma}_{1,0} = \frac{a_1}{\Delta_x}$$

(a.k.a. kill the diffusion, one sided drift)

- flat solution: $\boldsymbol{u(x_0)} = u(x_1)$

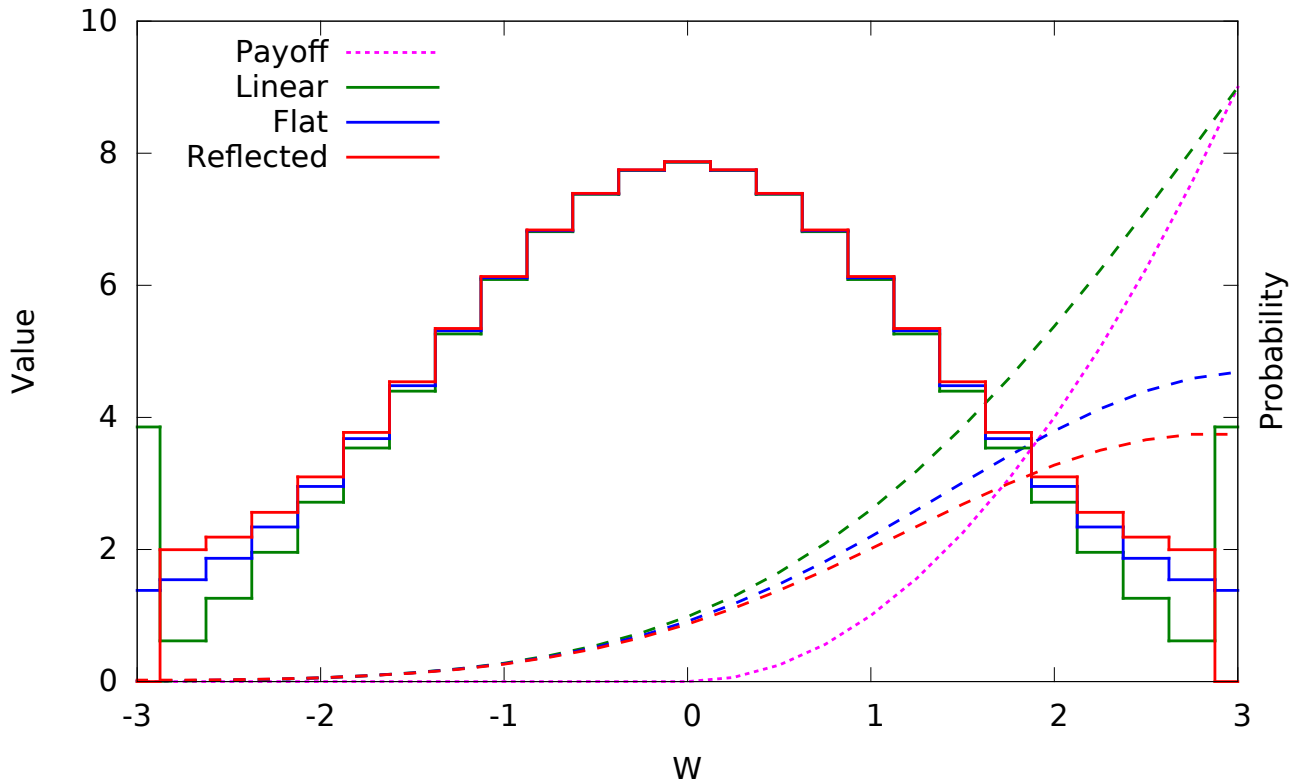
$$\gamma_{1,2} = \hat{\gamma}_{1,2} = \frac{b_1^2}{2\Delta_x^2} + \frac{a_1}{2\Delta_x}$$

(a.k.a. just don't jump outside)

Boundary conditions

Example

- W_t is a standard Brownian Motion
- payoff: $\max(W_2, 0)^2$
- grid: 25 points, $W_t \in [-3, 3]$



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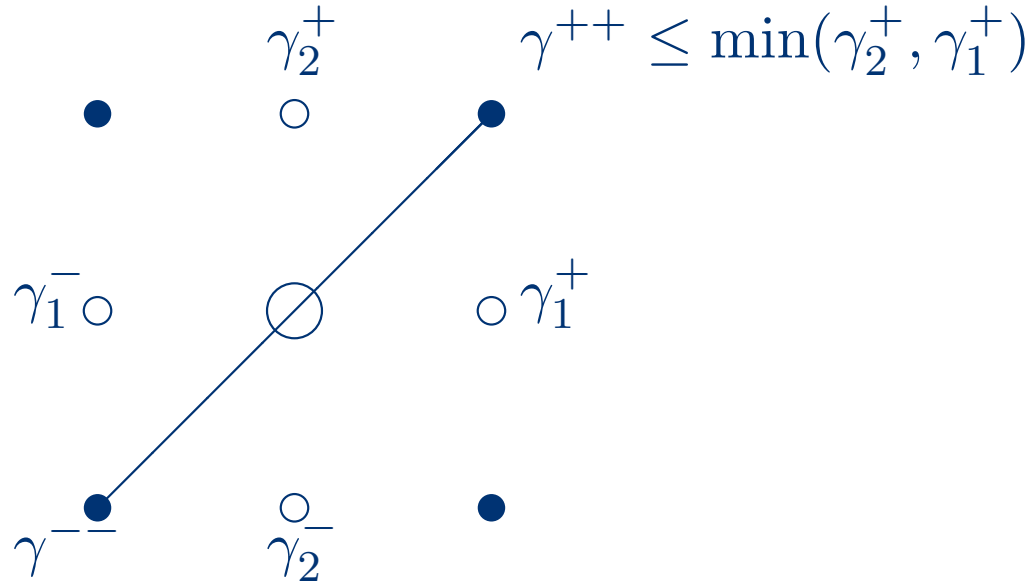
Correlation

Only the corners contribute to the cross product ($\rho > 0$)

$$\begin{aligned}\frac{1}{h}\mathbb{E}\left[\Delta X_h^1 \Delta X_h^2\right] &= \rho b_1 b_2 \\ &= \gamma^{++}\Delta_1^+\Delta_2^+ + \gamma^{--}\Delta_1^-\Delta_2^- \\ &- \gamma^{+-}\Delta_1^+\Delta_2^- - \gamma^{-+}\Delta_1^-\Delta_2^+\end{aligned}$$



hard to keep all intensities positive.

Correlation

Positive correlation

The maximum *positive* correlation that can be achieved (without introducing negative intensities) is

$$\frac{1}{b_1 b_2}\left(\min(\gamma_1^+, \gamma_2^+)\Delta_1^+\Delta_2^+ + \min(\gamma_1^-, \gamma_2^-)\Delta_1^-\Delta_2^-\right) \leq 1$$

If the scale in the 2 dimensions is similar ($\gamma_1 \sim \gamma_2$), a wide range of correlations can be attained.

One would need an *exponentially fitted* scheme from cross terms (does it exist?)

Forward equation: mass or density?

- As the state space of a CTMC is finite, it is natural to describe distributions by actual *probabilities*
- Can we use densities ($\mathbf{w}$) instead?
- Yes, but we need a different semigroup

$$\begin{aligned} \Delta_X &= \operatorname{diag}(\Delta_{x_i}) \\ \mathbf{w} &= \Delta_X^{-1}\mathbf{y} \\ \sum \mathbf{w}_i \Delta_{x_i} &= 1 \\ \mathbf{w}_s &= \left[\Delta_X^{-1} e^{s\Gamma'} \Delta_X\right] \mathbf{w}_0 \\ &\neq e^{s\Gamma'}\mathbf{w}_0 \end{aligned}$$

unless the grid is uniform

Lognormal process

Uniform in log, or logarithmic in the asset

How to choose the grid for a lognormal process?

- uniform discretisation of the logarithmic process (normal)
- or, logarithmic discretisation of the lognormal process

If we select the correct boundary conditions (loglinear vs linear), the Γ matrices will be almost the same.
A direct discretisation of S_t has the advantage of matching *by construction* the mean of the process (not just in the limit)

$$\begin{aligned}\mathbb{E}^i\left[e^{X_h} - e^{X_0}\right] &\sim \left(\gamma^-(e^{-\Delta_x} - 1) + \gamma^+(e^{\Delta_x} - 1)\right) e^{x_i} h \\ &\sim -\frac{1}{24} b_i^2 \Delta_x^2 e^{x_i} h\end{aligned}$$

an error we can avoid for free.

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